Energy-Related Controllability of Signed Complex Networks With Laplacian Dynamics

Baike She, Siddhartha Mehta, Chau Ton, and Zhen Kan

***Abstract*—This article investigates energy-related controllability of complex networks. Specifically, our objective is to establish controllability characteristics on signed complex networks, where the network units interact via neighbor-based Laplacian feedback and the network admits positive and negative edges to capture cooperative and competitive interactions among these units. The network units can be classified into leaders and followers. This article focuses on characterizing the energy-related controllability in signed networks (i.e., the energy incurred by the leaders in the control of a network). To this end, controllability Gramian-based measures are exploited to quantify the difficulty of the control problem on signed networks in terms of the required control energy. Fundamental relationships between these measures and network topology are developed via graph Laplacian to characterize energy-related controllability. It is revealed that, for structurally unbalanced signed graphs, the energy-related controllability is closely related to the diagonal entries of the inverse of the graph Laplacian. It is also discovered that structurally balanced signed graphs and their corresponding unsigned graphs have the same energy-related controllability.**

***Index Terms*—Control energy, network controllability, signed networks.**

I. Introduction

Complex networks composed of units dynamically interacting among themselves have found broad applications in brain networks, social networks, multiagent networks, and power networks. In such applications, the ability to steer a network to a desired behavior via external controls, referred to as network controllability, is of fundamental significance to realize system functionalities. One popular approach is to cast the network control problem into a leader–follower framework, wherein the leaders dictate the overall behavior of the network by influencing the followers via the connectivity characteristics of the network. Investigation into finding leader groups that can render the network controllable has generated a substantial research volume. Structural controllability [1]–[3], graph-theoretic approaches [4]–[6], and topological properties [7]–[9], among others, were extensively explored to facilitate leader group selection for network controllability. However, along with controllability, an important aspect, stemming from practical applicability, that needs to be considered is the energy needed to control a network. This is particularly critical in situations, where the network might be controllable by a selected leader group, but the control energy required by the leaders might be infeasible to allocate in practice. To this effect, this article is motivated by the desire to investigate the energy-related controllability of signed leader–follower complex networks.

Various metrics of network controllability have been developed to characterize control energy, among which the most widely used is the controllability Gramian, whose structure relates to the energy notions of the network controllability. The properties of controllability Gramian, such as its minimum eigenvalue, the trace of its inverse, and the condition number, have been extensively explored in the works of [10]–[12] to characterize the energy-related performance in network control. Control energy can also be characterized via spectral analysis of system matrices, which have been found to hold a relationship with the controllability Gramian. In [13], it was discovered that the minimal control energy is related to the distribution of the eigenvalues of the system matrix. In [14], the leading right and left eigenvectors of the system matrix were found to play a crucial role in quantifying how much each node contributes to the network in terms of controllability and control energy. Other representative approaches include optimization-based leader group selection for minimal control energy [15]–[17], graph-theoretical characterizations of energy-constrained controllability [18]–[20], and network design methods for improved controllability and energy efficiency [21]. Such recent advances in network science have provided control formalisms that include energy considerations to derive practically feasible solutions to multiagent control systems. In contrast to existing results, the present article investigates energy-related controllability of signed graphs with Laplacian dynamics.

Motivated by recent advances, this article investigates the energy-related controllability of signed undirected networks, where the network units interact via neighbor-based Laplacian feedback, and the network allows positive and negative edges to capture cooperative and competitive interactions among network units. The network units are classified as either leaders or followers. The energy-related controllability jointly considers network controllability (i.e., the ability to drive the network to a desired state by a leader group) and energy requirements (i.e., the control energy incurred by the selected leaders in steering the network to the desired state). Specifically, energy-related measures, namely, average controllability, average control energy, and volumetric control energy, are considered. These measures are then characterized in relation to signed graph Laplacian to gain topological insights into the energy-related controllability of complex networks.

The contributions of this article are multifold. This work relates the energy-related metrics to signed undirected networks with Laplacian dynamics. It is revealed that the inverse signed graph Laplacian can be used to quantify how the leaders individually contribute to network control in terms of energy-related controllability. Nodal centrality (i.e., a measure of individual contributions of nodes in network control) was

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Baike She is with the Department of Mechanical Engineering, The University of Iowa, Iowa City, IA 52246 USA (e-mail: baikeshe@uiowa.edu).

Siddhartha Mehta is with the Department of Industrial and Systems Engineering, University of Florida Research and Engineering Education Facility, Shalimar, FL 32579 USA (e-mail: siddhart@ufl.edu).

Chau Ton is with the Southwest Research Institute, San Antonio, TX 78238 USA (e-mail: chau.t.ton@gmail.com).

Zhen Kan is with the Department of Automation, University of Science and Technology of China, Hefei 230052, China (e-mail: zkan@ustc.edu.cn).

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previously investigated in the works of [11], [12], and [22]–[27]. In [22], network vulnerability was investigated in terms of the minimum average control energy required for the adversary to drive the system away from synchronization. Similar energy metrics were also approached independently based on the concept of submodularity [11], the joint centrality measure for reduced control energy [23], the nodal communicability to support actuator selection [24], [25], and the reachability metrics for bilinear networks [26]. A common approach employed in the aforementioned results is the exploitation of the controllability Gramian-based energy metrics. Aligned with these efforts, this article moves forward in relating the controllability Gramian-based energy metrics to the graph Laplacian. Different from the discrete systems (see, e.g., [12], [23], and [26]) or single leader cases (see, e.g., [22], [24], and [25]) in the literature, this article considers complex networks evolving with continuous signed Laplacian dynamics and multiple leaders. Since the graph Laplacian is a global topological property of a network, the developed characterizations reveal how the energy-related controllability is influenced by the network topology. It is revealed that, for structurally unbalanced signed graphs, the energy-related controllability is closely related to the diagonal entries of the inverse of the graph Laplacian, which provides insights into how much an individual leader contributes to the network's overall energy expenditure. Since the inverse graph Laplacian can be interpreted as the graph resistance [28], the revealed relation can be potentially leveraged to characterize energy-related controllability from topological perspectives via graph resistances. It is also discovered that structurally balanced signed graphs and their corresponding unsigned graphs have the same energy-related controllability, which indicates that the existing results developed for unsigned networks are applicable to structurally balanced signed networks. In addition, the developed topological characterizations of the energy-related controllability are generic, in the sense that they not only hold for signed graphs, but also for unsigned graphs, since unsigned graphs are a particular case of signed graphs that only consider positive edges.

II. Problem Formulation

Consider a complex network represented by an undirected signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where the node set $\mathcal{V} = \{v_1, \ldots, v_n\}$ and the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ represent the network units and their interactions, respectively. The network-wide interactions are captured by the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ij} \neq 0$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. No self-loop is considered, i.e., $a_{ii} = 0 \forall i = 1, \ldots, n$. Different from unsigned graphs that exclusively contain nonnegative adjacency matrices, $a_{ij} : \mathcal{E} \to \mathbb{R}$ in this article is allowed to admit positive or negative weights to capture collaborative or competitive interactions between network units, thus resulting in a signed graph $\mathcal{G}$. Let $d_i = \sum_{j\in\mathcal{N}_i} |a_{ij}|$, where $\mathcal{N}_i = \{v_j | (v_i, v_j) \in \mathcal{E}\}$ denotes the neighbor set of v_i and $|a_{ij}|$ denotes the absolute value of a_{ij}. The graph Laplacian of $\mathcal{G}$ is defined as $\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D} - \mathcal{A}$, where $\mathcal{D} \triangleq \text{diag}\{d_1, \ldots, d_n\}$ is a diagonal matrix. Since $\mathcal{G}$ is undirected, the graph Laplacian $\mathcal{L}(\mathcal{G})$ is symmetric.

Let $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ denote the stacked system states,[1] where each entry $x_i(t) \in \mathbb{R}$ represents the state of node v_i. Suppose the system states evolve according to the following Laplacian dynamics:

$$\dot{x}(t) = -\mathcal{L}(\mathcal{G})\, x(t) \tag{1}$$

where the graph Laplacian $\mathcal{L}(\mathcal{G})$ indicates that each node updates its state taking into account the states of its neighboring nodes.

It is assumed that a set $\mathcal{K} = \{v_{k_1}, \ldots, v_{k_m}\} \subseteq \mathcal{V}$, $|\mathcal{K}| = m \geq 1$, of nodes, referred as leaders in the network, can be endowed with external controls. With external inputs, the system dynamics in (1) can be rewritten as

$$\dot{x}(t) = -\mathcal{L}(\mathcal{G})\, x(t) + B_{\mathcal{K}} u(t) \tag{2}$$

where $B_{\mathcal{K}} = [\, e_{k_1} \cdots e_{k_m} \,] \in \mathbb{R}^{n\times m}$ is the input matrix with basis vectors e_i, $i = k_1, \ldots, k_m$, indicating that the ith node is endowed with external controls $u(t) \in \mathbb{R}^m$. The dynamics of (2) indicates that the network behavior is not only driven by the graph Laplacian $\mathcal{L}$, but also depends on the input matrix $B_{\mathcal{K}}$ via the leader set $\mathcal{K}$. Different leader sets can result in different $B_{\mathcal{K}}$, leading to drastic differences in the capability of controlling a network, which is elucidated by introducing the definition of leader–follower controllability. The network model with Laplacian dynamics as in (2) has various applications in many engineering systems (cf. [5], [7], [22], and [29]).

A network with dynamics in (2) is controllable, if the controllability matrix $\mathcal{C}_{\mathcal{K}} = [B_{\mathcal{K}} - \mathcal{L}B_{\mathcal{K}} \cdots (-1)^n \mathcal{L}^n B_{\mathcal{K}}]$ has a full row rank. Hence, in theory, a network can be controllable with appropriate selection of leader nodes (and consequently $B_{\mathcal{K}}$). However, it does not tell how difficult it is to control the network in practice, i.e., how much energy is needed to drive the network to the target state. To provide energy-related quantification of network control, the total control energy over the time interval $[0, t]$ is given by

$$E(t) = \int_0^t \|u(\tau)\|_2^2\, d\tau \tag{3}$$

where $\|u\|_2$ represents the Euclidean norm of u. Assuming the initial state $x(0) = 0$ and the optimal control $u(t)$ in [30], the minimum control energy required to drive the system in (2) from $x(0)$ to a desired target state x_f is

$$E(t) = x_f^T \mathcal{W}_{\mathcal{K}}^{-1}(t) x_f \tag{4}$$

where

$$\mathcal{W}_{\mathcal{K}}(t) = \int_0^t e^{-\mathcal{L}\tau} B_{\mathcal{K}} B_{\mathcal{K}}^T e^{-\mathcal{L}^T\tau} d\tau \tag{5}$$

is the controllability Gramian at time t, which is positive definite if and only if the system in (2) is leader–follower controllable.

Since the controllability Gramian $\mathcal{W}_{\mathcal{K}}$ provides an energy-related measure of network controllability, various quantitative metrics of controllability were developed based on $\mathcal{W}_{\mathcal{K}}$. As discussed in [11], the trace $\text{tr}(\mathcal{W}_{\mathcal{K}})$ provides an overall measure of network controllability in all directions. The average control energy required to move the system in (2) to a target state is obtained as the trace $\text{tr}(\mathcal{W}_{\mathcal{K}}^{-1})$. The volumetric control energy given by $\log(\det \mathcal{W}_{\mathcal{K}})$ measures the volume of the ellipsoid containing the target states that can be reached with unit control input. The objective of this article is to establish these metrics in the context of signed networks and develop topological characterizations on how these metrics are related to the graph Laplacian in quantifying the energy expenditure in the control of signed networks.

III. Energy-Related Controllability of Signed Networks

Based on the topological structures, signed graphs can be classified as either structurally balanced or structurally unbalanced.

Definition 1 (Structural balance): A signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced if the node set $\mathcal{V}$ can be partitioned into $\mathcal{V}_1$ and $\mathcal{V}_2$

[1] Generalizations to multidimensional system states (e.g., $x_i \in \mathbb{R}^m$) are expected to be trivial via the matrix Kronecker product.

with $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, where $a_{ij} > 0$ if $v_i, v_j \in \mathcal{V}_q$, $q \in \{1,2\}$, and $a_{ij} < 0$ if $v_i \in \mathcal{V}_q$ and $v_j \in \mathcal{V}_r$, $q \neq r$, and $q, r \in \{1,2\}$.

Definition 1 indicates that v_i and v_j are positive neighbors if they are from the same subset, i.e., either $\mathcal{V}_1$ or $\mathcal{V}_2$, and negative neighbors if v_i and v_j are from different subset. To characterize structural balance, necessary and sufficient conditions are provided as follows.

Lemma 1 (see [31]): A connected signed graph $\mathcal{G}$ is structurally balanced if and only if any of the following equivalent conditions holds.

1) All cycles[2] of $\mathcal{G}$ are positive, i.e., the product of edge weights on any cycle is positive.
2) There exists a diagonal matrix $\Phi = \text{diag}\{\phi_1, \ldots, \phi_n\} \in \mathbb{R}^{n\times n}$ with $\phi_i \in \{\pm 1\}$ such that $\Phi\mathcal{A}\Phi \in \mathbb{R}^{n\times n}$ has nonnegative entries.
3) 0 is an eigenvalue of graph Laplacian $\mathcal{L}(\mathcal{G})$.

Since $\mathcal{G}$ is a signed graph, its graph Laplacian $\mathcal{L}(\mathcal{G})$ may have negative off-diagonal entries and its row/column sums are not necessarily zero, which indicates that 0 is no longer a default eigenvalue as in the case of unsigned graphs. Lemma 1 (condition 3) further indicates that $\mathcal{L}(\mathcal{G})$ of a structurally balanced graph is singular (i.e., contains eigenvalue 0), while $\mathcal{L}(\mathcal{G})$ of a structurally unbalanced graph is nonsingular. Therefore, to develop controllability results on signed graphs, the subsequent development will consider the cases when the graph is structurally unbalanced and balanced. When considering structurally unbalanced signed graphs in Section III-A, we will focus on the infinite-horizon Gramian, i.e., the case when $t \to \infty$ in (5), due to asymptotic or exponential convergence/stability of most dynamical systems. Since the Gramian $\mathcal{W}_\mathcal{K}$ in (5) can be unbounded as $t \to \infty$ when considering structurally balanced signed graphs, the Gramian matix over a finite-time interval $[t_0, t_f]$ is considered in Section III-B.

A. Structurally Unbalanced Signed Graphs

Controllability Gramian-based metrics are widely used to characterize the energy required in network control. However, the eigenproperties of the Gramian are typically challenging to characterize analytically. In addition, topological characterizations of the energy-related controllability are hard to be extracted from the Gramian matrix. To overcome this issue, inspired by the Gramian-based nodal centrality, e.g., in [11], [12], [22], [23], and [25], we introduce a notion termed *nodal metric* to quantify how much each node contributes to the energy-related controllability of structurally unbalanced signed graphs, which is defined based on the signed graph Laplacian.

Definition 2 (Nodal metric): Consider a structurally unbalanced signed network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with graph Laplacian $\mathcal{L}(\mathcal{G})$. Let $\mathcal{M}(\cdot): \mathcal{V} \to \mathbb{R}$ denote a metric associated with node v_i, which is defined as the ith diagonal entry of the inverse of the graph Laplacian $\mathcal{L}(\mathcal{G})$, i.e., $\mathcal{M}(v_i) = \mathcal{L}^{-1}_{ii}(\mathcal{G})$.

Based on the nodal metric defined in Definition 2, the following development determines how various energy-related controllability measures, i.e., average controllability (see Theorem 1), average control energy (see Theorem 2), and volumetric control energy (see Theorem 3), are related to the total nodal metric of the control nodes (i.e., leaders) via graph Laplacian.

Theorem 1 (Average controllability): Consider an undirected signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ evolving according to the dynamics in (2) with the leader set $\mathcal{K}$. If $\mathcal{G}$ is structurally unbalanced, the average controllability $\text{tr}(\mathcal{W}_\mathcal{K})$ with the controllability Gramian $\mathcal{W}_\mathcal{K}$ defined in (5) can be characterized by the sum of the total $\mathcal{M}$ of the leaders in $\mathcal{K}$ as

$$\text{tr}\,(\mathcal{W}_\mathcal{K}) = \frac{1}{2}\sum_{i\in\mathcal{K}} \mathcal{M}\,(v_i).$$

Proof: If $\mathcal{G}$ is structurally unbalanced, i.e., its graph Laplacian $\mathcal{L}(\mathcal{G})$ does not have a zero eigenvalue according to Lemma 1, then $\mathcal{L}(\mathcal{G})$ is a symmetric positive-definite matrix. Let $\lambda_j \in \mathbb{R}$ and $p_j = [\,p_{j1} \cdots p_{jn}\,]^T \in \mathbb{R}^n$, $j \in \{1, \ldots, n\}$, be the eigenvalues and the corresponding normalized eigenvectors of $\mathcal{L}(\mathcal{G})$, respectively. Then, the graph Laplacian can be written as $\mathcal{L} = P\varLambda P^T$, where $\varLambda = \text{diag}\{\lambda_1, \ldots, \lambda_n\} \in \mathbb{R}^{n\times n}$ is a diagonal matrix and $P = [\,p_1 \cdots p_n\,] \in \mathbb{R}^{n\times n}$. Using the fact that

$$e^{-\mathcal{L}\tau} = e^{-P\varLambda P^T\tau} = Pe^{-\varLambda\tau}P^T$$

the infinite-horizon controllability Gramian $\mathcal{W}_\mathcal{K}$ of the system in (2) is given by

$$\mathcal{W}_\mathcal{K} = P\left(\int_0^\infty e^{-\Lambda\tau}P^TB_\mathcal{K}B_\mathcal{K}^TPe^{-\Lambda\tau}d\tau\right)P^T. \quad (6)$$

Since the trace is invariant under cyclic permutations, the trace of $\mathcal{W}_\mathcal{K}$ is obtained from (6) as

$$\begin{aligned}\text{tr}\,(\mathcal{W}_\mathcal{K}) &= \text{tr}\left(\int_0^\infty e^{-\Lambda\tau}P^TB_\mathcal{K}B_\mathcal{K}^TPe^{-\Lambda\tau}d\tau P^TP\right)\\ &= \text{tr}\left(\int_0^\infty e^{-\Lambda\tau}P^TB_\mathcal{K}B_\mathcal{K}^TPe^{-\Lambda\tau}d\tau\right) \quad (7)\end{aligned}$$

where $P^TP = I_n$ is used with I_n denoting an $n \times n$ identity matrix. Substituting P and $B_\mathcal{K}$ into (7) yields

$$\text{tr}\,(\mathcal{W}_\mathcal{K}) = \int_0^\infty e^{-2\lambda_1\tau}\sum_{i\in\mathcal{K}}p_{1i}^2 + \cdots + e^{-2\lambda_n\tau}\sum_{i\in\mathcal{K}}p_{ni}^2 d\tau$$

which can be reorganized as

$$\text{tr}\,(\mathcal{W}_\mathcal{K}) = \int_0^\infty \sum_{i=1}^n e^{-2\lambda_i\tau}p_{ik_1}^2 + \cdots + \sum_{i=1}^n e^{-2\lambda_i\tau}p_{ik_m}^2 d\tau \quad (8)$$

where $k_1, \ldots, k_m$ are the leader indices as defined in (2). Since $\int_0^\infty e^{-2\lambda_i\tau}d\tau = \frac{1}{2\lambda_i}$, (8) can be further simplified into

$$\begin{aligned}\text{tr}\,(\mathcal{W}_\mathcal{K}) &= \sum_{i=1}^n \frac{1}{2\lambda_i}p_{ik_1}^2 + \cdots + \sum_{i=1}^n \frac{1}{2\lambda_i}p_{ik_m}^2\\ &= \sum_{j\in\mathcal{K}}\sum_{i=1}^n \frac{1}{2\lambda_i}p_{ij}^2. \quad (9)\end{aligned}$$

Since $\mathcal{L}^{-1} = P\varLambda^{-1}P^T$, it is always true that

$$\mathcal{L}_{jj}^{-1} = \sum_{i=1}^n \frac{1}{\lambda_i}p_{ij}^2 \quad (10)$$

which indicates from (9) that $\text{tr}(\mathcal{W}_\mathcal{K}) = \frac{1}{2}\sum_{j\in\mathcal{K}}\mathcal{L}_{jj}^{-1}$, where $\mathcal{L}_{jj}^{-1}$ is the jth diagonal entry of $\mathcal{L}^{-1}$. Therefore, by Definition 2, $\text{tr}(\mathcal{W}_\mathcal{K}) = \frac{1}{2}\sum_{i\in\mathcal{K}}\mathcal{M}(v_i)$. ∎

As discussed in [11], $\text{tr}(\mathcal{W}_\mathcal{K})$ can be interpreted as the average controllability, providing an overall measure of the network controllability. If a network with dynamics in (2) is uncontrollable in certain direction of its state space, then the eigenvalue of $\mathcal{W}_\mathcal{K}$ corresponding to that direction will be zero, resulting in infinite control energy in that direction due to the term $\mathcal{W}_\mathcal{K}^{-1}$ in (4). Likewise, if the system is controllable in certain direction but with a small eigenvalue, then higher energy can be expected to be required to control the network in that direction. Consequently, as the sum of eigenvalues of $\mathcal{W}_\mathcal{K}$, $\text{tr}(\mathcal{W}_\mathcal{K})$ characterizes

[2] A cycle is composed of concatenated distinct edges $\{(v_1v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\} \subset \mathcal{E}$, where the starting and end nodes are identical, i.e., $v_1 = v_k$.

the average difficulty of network control in all directions. In addition, $\mathrm{tr}(\mathcal{W}_\mathcal{K})$ is also closely related to the H_2 norm of the system. It is shown in [32] and [33] that the system robustness to external disturbances can be improved based on optimizing $\mathrm{tr}(\mathcal{W}_\mathcal{K})$. As discussed in [34], the diagonal entries of the unsigned Laplacian matrices corresponding to the leaders can be used to indicate the average controllability of an unsigned graph. Theorem 1 extends this result to signed graphs and reveals that the signed graph Laplacian is the key to determine the average controllability of a signed network. Specifically, the diagonal entries of the inverse graph Laplacian corresponding to the selected leaders together determine the average difficulty of network control.

Remark 1: Although leader selection for traditional network controllability has been studied in the literature, the selection of leaders that jointly considers network controllability and energy efficiency is generally a challenging and computationally hard combinatorial problem [17], [23], [35], [36]. A potential benefit of Theorem 1 is that it provides constructive insights, as well as a quantitative method, in selecting leader groups for energy-efficient network control. For instance, existing methods (cf. [37] and [38] to name a few) can be first used to identify a set of different leader groups that can render leader–follower controllability. Since a greater value of average controllability generally indicates improved network controllability and better H_2 performance, the candidate groups can be further refined based on Theorem 1 by selecting a leader group $\mathcal{K}$ with a greater value of $\sum_{i\in\mathcal{K}}\mathcal{M}(v_i)$ for energy-efficient network control, where $\mathcal{M}(v_i)$ can be simply determined from the graph Laplacian.

Theorem 2 (Average control energy): Consider an undirected signed graph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ evolving according to the dynamics in (2). If $\mathcal{G}$ is structurally unbalanced and the system (2) under the leader group $\mathcal{K}$ is controllable, the average control energy metric $\mathrm{tr}(\mathcal{W}_\mathcal{K}^{-1})$ can be lower bounded by the total $\mathcal{M}$ of the leaders in $\mathcal{K}$ as

$$\mathrm{tr}\left(\mathcal{W}_\mathcal{K}^{-1}\right)\geq\frac{2n^2}{\sum_{i\in\mathcal{K}}\mathcal{M}\left(v_i\right)}.$$

Proof: Let $\gamma_i\in\mathbb{R}$ and $q_i=[\,q_{i1}\ \cdots\ q_{in}\,]^T\in\mathbb{R}^n, i\in\{1,\ldots,n\}$, be the eigenvalues and the corresponding normalized eigenvectors of $\mathcal{W}_\mathcal{K}$, respectively. The trace of $\mathcal{W}_\mathcal{K}$ and $\mathcal{W}_\mathcal{K}^{-1}$ are $\mathrm{tr}(\mathcal{W}_\mathcal{K})=\sum_{i=1}^n\gamma_i$ and $\mathrm{tr}(\mathcal{W}_\mathcal{K}^{-1})=\sum_{i=1}^n\frac{1}{\gamma_i}$, respectively. Provided that the leader group $\mathcal{K}$ can render the system leader–follower controllable, i.e., $\gamma_i>0,\forall i$, the arithmetic mean will not be less than the harmonic mean, i.e., $\frac{1}{n}\sum_{i=1}^n\gamma_i\geq n(\sum_{i=1}^n\frac{1}{\gamma_i})^{-1}$. Therefore, similar to [12], Theorem 1 yields

$$\mathrm{tr}\left(\mathcal{W}_\mathcal{K}^{-1}\right)\geq\frac{n^2}{\mathrm{tr}\left(\mathcal{W}_\mathcal{K}\right)}=\frac{2n^2}{\sum_{i\in\mathcal{K}}\mathcal{M}\left(v_i\right)}.$$

■

Since the control energy $E(t)$ in (4) is proportional to $\mathcal{W}_\mathcal{K}^{-1}$, $\mathrm{tr}(\mathcal{W}_\mathcal{K}^{-1})$ measures the average energy needed to control a network to an arbitrary target state. Theorem 2 indicates that $\mathrm{tr}(\mathcal{W}_\mathcal{K}^{-1})$ is inversely proportional to the total $\mathcal{M}$ of the leader group $\mathcal{K}$, i.e., $\sum_{i\in\mathcal{K}}\mathcal{M}(v_i)$. In other words, selecting a leader group with higher total $\mathcal{M}$ results in less energy expenditure in network control. The comparison between average controllability and volumetric control energy of discrete systems was considered in [12]. An upper bound of average control energy was derived in [22] but with a focus on single leader–follower unsigned networks with distinct eigenvalues. In contrast, Theorem 2 generalizes results to signed networks with continuous Laplacian dynamics.

Theorem 3 (Volumetric control energy): Consider an undirected signed graph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ evolving according to the dynamics in (2). If $\mathcal{G}$ is structurally unbalanced, the volumetric control energy metric $\log(\det\mathcal{W}_\mathcal{K})$ can be upper bounded by the total $\mathcal{M}$ of the leaders in $\mathcal{K}$ as

$$\log\left(\det\mathcal{W}_\mathcal{K}\right)\leq\sum_{i\in\mathcal{K}}n\log\left(\frac{\mathcal{M}\left(v_i\right)}{n}\right)+c'_\Psi$$

where $c'_\Psi\in\mathbb{R}^+$ is a constant determined by the eigenvalues of the graph Laplacian.

Proof: The proof starts by characterizing $\log(\det\mathcal{W}_\mathcal{K})$ for the case of a single leader in the network, which will then be extended to a multileader scenario.

Consider the case of a single leader, i.e., $m=1$, and let the leader's index be k_m. As a result, the input matrix $B_\mathcal{K}$ is reduced to a basis vector $e_{k_m}\in\mathbb{R}^n$, where the k_mth entry is one, while the others are zeros. The controllability Gramian $\mathcal{W}_\mathcal{K}$ for $\mathcal{K}=\{k_m\}$ can be simplified from (6) as

$$\mathcal{W}_\mathcal{K}=P\left(\int_0^\infty e^{-\Lambda\tau}P^Te_{k_m}e_{k_m}^TPe^{-\Lambda\tau}d\tau\right)P^T. \tag{11}$$

From the definition of P, $P^Te_{k_m}$ can be written as $P^Te_{k_m}=[\,p_{1k_m},\ \ldots,\ p_{nk_m}\,]^T\in\mathbb{R}^n$. Using the fact that $e^{-\Lambda\tau}\in\mathbb{R}^{n\times n}$ is a diagonal matrix, one can obtain

$$e^{-\Lambda\tau}P^Te_{k_m}=P_{k_m}z \tag{12}$$

where $P_{k_m}\triangleq\mathrm{diag}\{\,p_{1k_m},\ \ldots,\ p_{nk_m}\,\}\in\mathbb{R}^{n\times n}$ is a diagonal matrix, and $z=[\,e^{-\lambda_1\tau},\ \ldots,\ e^{-\lambda_n\tau}\,]^T\in\mathbb{R}^n$. Substituting (12) into (11), $\mathcal{W}_\mathcal{K}$ can be rewritten as

$$\begin{aligned}\mathcal{W}_\mathcal{K}&=PP_{k_m}\left(\int_0^\infty zz^Td\tau\right)P_{k_m}^TP^T\\&=PP_{k_m}\Psi P_{k_m}^TP^T\end{aligned} \tag{13}$$

where $\Psi=[\Psi_{ij}]\in\mathbb{R}^{n\times n}$ with $\Psi_{ij}=\frac{1}{\lambda_i+\lambda_j}$. Since P, P_{k_m}, and Ψ are all square matrices, the determinant of $\mathcal{W}_\mathcal{K}$ is

$$\begin{aligned}\det\left(\mathcal{W}_\mathcal{K}\right)&=\det P\det P_{k_m}\det\Psi\det P_{k_m}^T\det P^T\\&=\det\left(P_{k_m}P_{k_m}^T\right)\det\left(\Psi\right)\end{aligned} \tag{14}$$

where the fact that $\det P\det P^T=1$ is used since P is an orthogonal matrix. Based on (14), the volumetric control energy can be obtained as

$$\log\left(\det\mathcal{W}_\mathcal{K}\right)=\log\left(\det\left(P_{k_m}P_{k_m}^T\right)\right)+c_\Psi \tag{15}$$

where $c_\Psi\triangleq\log(\det\Psi)$ is a constant determined by the eigenvalues of $\mathcal{L}(\mathcal{G})$, and P_{k_m} is determined by the selection of the leader node $k_m\subseteq\mathcal{V}$. Since P_{k_m} is a diagonal matrix, it follows from (15) that

$$\log\left(\det\mathcal{W}_\mathcal{K}\right)=\log\left(\prod_{i=1}^np_{ik_m}^2\right)+c_\Psi.$$

Multiplying the above expression by $\frac{1}{n}$ on both sides, we get

$$\begin{aligned}\frac{1}{n}\log\left(\det\mathcal{W}_\mathcal{K}\right)&=\log\left(\prod_{i=1}^np_{ik_m}^2\right)^{\frac{1}{n}}+\frac{c_\Psi}{n}\\&\leq\log\frac{\sum_{i=1}^np_{ik_m}^2}{n}+\frac{c_\Psi}{n}\end{aligned} \tag{16}$$

where the fact that $(\prod_{i=1}^np_{ik_m}^2)^{\frac{1}{n}}\leq\frac{\sum_{i=1}^np_{ik_m}^2}{n}$ is used. Subtracting $\log(\lambda_{max})$ on both sides of (16) yields

$$\log\left(\det\mathcal{W}_\mathcal{K}\right)-n\log\left(\lambda_{max}\right)\leq n\log\frac{\sum_{i=1}^np_{ik_m}^2}{n\lambda_{max}}+c_\Psi. \tag{17}$$

In (17), $\lambda_{\max}>0$ is the largest eigenvalue or spectral radius of $\mathcal{L}(\mathcal{G})$. Furthermore, the expression in (10) yields $\sum_{i=1}^n\frac{1}{\lambda_{\max}}p_{ik_m}^2\leq$

$\sum_{i=1}^{n} \frac{1}{\lambda_i} p_{ik_m}^2 = \mathcal{L}_{k_m k_m}^{-1}$. Using this expression, the inequality in (17) can be written as

$$\log\left(\det \mathcal{W}_{\mathcal{K}}\right) \leq n \log\left(\frac{\mathcal{L}_{k_m k_m}^{-1}}{n}\right) + c'_{\Psi} \tag{18}$$

where $c'_{\Psi} \triangleq c_{\Psi} + n\log(\lambda_{\max})$.

For the multileader case (i.e., $m > 1$), let $\log(\det \mathcal{W}_{\mathcal{K}_i})$ denote the volumetric control energy associated with the leader $v_i \in \mathcal{K}$. From (18), one has

$$\log\left(\det \mathcal{W}_{\mathcal{K}_i}\right) \leq n \log \frac{\mathcal{M}\left(v_i\right)}{n} + c'_{\Psi}.$$

As demonstrated in [11, Sec. D, Th. 6], since $\log(\det \mathcal{W}_{\mathcal{K}})$ is a submodular function on the leader set, we have

$$\begin{aligned}\log\left(\det \mathcal{W}_{\mathcal{K}}\right) &\leq \sum_{i\in\mathcal{K}} \log\left(\det \mathcal{W}_{\mathcal{K}_i}\right) \\ &\leq \sum_{i\in\mathcal{K}} \left(n \log \frac{\mathcal{M}\left(v_i\right)}{n} + c'_{\Psi}\right)\end{aligned}$$

where $\log(\det \mathcal{W}_{\mathcal{K}})$ denotes total volumetric control energy of the system under the leader set $\mathcal{K}$. ■

The volumetric measure $\log(\det \mathcal{W}_{\mathcal{K}})$ indicates the set of states that can be reached with a unit control energy. Hence, a greater value of $\log(\det \mathcal{W}_{\mathcal{K}})$ usually implies a larger target space that can be reached with the same amount of control energy. Volumetric control energy was investigated in [11] and [39] using optimization and data-driven methods. In contrast, Theorem 3 reveals the relationship between the defined nodal metric and the volumetric controllability.

Remark 2: Theorems 1–3 not only show how various energy measures are related to the nodal metric of the leader group in the network, but also reveal that these measures are closely related to the inverse of the graph Laplacian. As such, these theorems offer a new perspective to design networks from the energy considerations. For the different controllability metrics, i.e., average controllability, average control energy, and volumetric control energy, it is shown that the respective metrics can be improved if the network topology is designed such that the diagonal entries of the inverse graph Laplacian corresponding to the leaders are maximized. In addition, as discussed in the works of [18], [28], and [40], the inverse of the graph Laplacian can be interpreted as the graph resistances, which play important roles in distributed network control and estimation. For instance, the graph resistances appear in network control problems, in which agents are steered toward a desired formation [40], and also appear in least-squares estimation problems, in which global information can be reconstructed from noisy measurements. In the recent work [18], the graph resistances were also used to quantify the information centrality. Based on Theorems 1–3, additional research will consider exploring how controllability metrics are related to the leader groups via graph resistances.

B. Structurally Balanced Signed Graph

This section considers the case that the signed graph $\mathcal{G}$ is structurally balanced. Lemma 1 indicates that, if $\mathcal{G}$ is structurally balanced, there always exists a gauge transformation Φ and a corresponding graph $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \bar{\mathcal{A}})$ with $\bar{\mathcal{A}} = [\bar{a}_{ij}] \in \mathbb{R}^{n\times n} = \Phi\mathcal{A}\Phi$. Clearly, $\bar{\mathcal{G}}$ is an unsigned correspondence of $\mathcal{G}$, i.e., they share the same node and edge sets except that the edge weights in $\bar{\mathcal{A}}$ are all nonnegative, i.e., $\bar{a}_{ij} = |a_{ij}|$.

Theorem 4: Consider a signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and its corresponding gauge transformed unsigned graph $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \bar{\mathcal{A}})$ with their controllability Gramians $\mathcal{W}_k$ and $\bar{\mathcal{W}}_k$, respectively. Let the nodes in $\mathcal{G}$ and $\bar{\mathcal{G}}$ evolve according to the dynamics in (2) but with their respective adjacency matrices $\mathcal{A}$ and $\bar{\mathcal{A}}$. If $\mathcal{G}$ is structurally balanced, then $\mathcal{W}_k$ has the same matrix spectrum as $\bar{\mathcal{W}}_k$.

Proof: For the unsigned graph $\bar{\mathcal{G}}$, its nodes evolve according to the following dynamics:

$$\dot{x}(t) = -\mathcal{L}_u\left(\mathcal{G}\right)x(t) + B_{\mathcal{K}}u(t) \tag{19}$$

where $\mathcal{L}_u = \mathcal{D} - \bar{\mathcal{A}}$ is the graph Laplacian of $\bar{\mathcal{G}}$. From (5), the controllability Gramian of (19) at time $t \in [t_0, t_f]$ can be obtained as

$$\bar{\mathcal{W}}_k(t) = \int_{t_0}^{t} e^{-\mathcal{L}_u\tau} B_{\mathcal{K}} B_{\mathcal{K}}^T e^{-\mathcal{L}_u^T\tau} d\tau.$$

From Lemma 1, if $\mathcal{G}$ is structurally balanced, then $\mathcal{L}_u = \mathcal{D} - \Phi(\mathcal{A})\Phi = \Phi\mathcal{L}\Phi$. Therefore, $\bar{\mathcal{W}}_k$ becomes

$$\begin{aligned}\bar{\mathcal{W}}_k(t) &= \int_{t_0}^{t} e^{-\Phi\mathcal{L}\Phi} B_{\mathcal{K}} B_{\mathcal{K}}^T e^{-\Phi\mathcal{L}^T\Phi\tau} d\tau \\ &= \int_{t_0}^{t} \Phi e^{-\mathcal{L}\tau}\Phi B_{\mathcal{K}} B_{\mathcal{K}}^T \Phi e^{-\mathcal{L}^T\tau}\Phi d\tau \\ &= \Phi\mathcal{W}_k\Phi\end{aligned} \tag{20}$$

where the fact $\Phi B_{\mathcal{K}} B_{\mathcal{K}}^T \Phi = B_{\mathcal{K}} B_{\mathcal{K}}^T$ is used, since gauge transformation preserves diagonal matrices, and $\mathcal{W}_{\mathcal{K}}(t) = \int_{t_0}^{t} e^{-\mathcal{L}\tau} B_{\mathcal{K}} B_{\mathcal{K}} e^{-\mathcal{L}\tau} d\tau$ is the controllability Gramian defined in (5). Since gauge transformation preserves the matrix spectra [31], the matrix $\Phi\mathcal{W}_k\Phi$ has the same set of eigenvalues as that of $\mathcal{W}_k$. Therefore, it is clear from (20) that $\mathcal{W}_k$ has the same matrix spectrum as of $\bar{\mathcal{W}}_k$. ■

The energy-related controllability metrics analyzed in Section III-A, i.e., average controllability $\mathrm{tr}(\bar{\mathcal{W}}_k)$, average control energy $\mathrm{tr}(\bar{\mathcal{W}}^{-1})$, and volumetric control energy $\log\det(\bar{\mathcal{W}}_k)$ along with other metrics, including the worst-case controllability $\lambda_{\min}(\bar{\mathcal{W}}_k)$ and the dimension of the controllable subspace $\mathrm{rank}(\bar{\mathcal{W}}_k)$, can be directly obtained from the eigenvalues of $\bar{\mathcal{W}}_k$. Since $\mathcal{W}_k$ and $\bar{\mathcal{W}}_k$ have the same set of eigenvalues, Theorem 4 indicates that the structurally balanced signed graph $\mathcal{G}$ and its corresponding unsigned graph $\bar{\mathcal{G}}$ are equivalent in terms of the above controllability metrics. Therefore, Theorem 4 provides a means to investigate the energy-related controllability of signed networks by examining its corresponding unsigned graph $\bar{\mathcal{G}}$ using the analysis and design methods developed for unsigned graphs.

Remark 3: When considering the case that $t \to \infty$, $\mathcal{W}_k(t)$ grows unboundedly with t due to the existence of the 0 eigenvalue. Since the zero eigenvalue indicates a consensus manifold, the infinite volume of the determinant of $\mathcal{W}_k(t)$ reflects that network state can be moved along the consensus manifold with arbitrarily small control energy. Therefore, in terms of the control energy required in moving network states, we only need to consider the manifolds corresponding to nonzero eigenvalues. The eigenvalue decomposition approach in [22] can then be leveraged to obtain a sub-Gramian that only contains nonzero eigenvalues, where similar analysis in Theorem 4 can be applied to obtain the equivalence of unsigned graph and signed structurally balanced graphs in terms of energy metrics.

Corollary 1: Consider a signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and its unsigned correspondence $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \bar{\mathcal{A}})$. If $\mathcal{G}$ is structurally balanced, then the leader–follower controllability of $\mathcal{G}$ is equivalent to that of $\bar{\mathcal{G}}$ under the same leader set.

It is well known that a system is controllable if and only if its controllability Gramian is positive definite. Thus, Corollary 1 follows immediately from Theorem 4 by the fact that $\mathcal{W}_k$ and $\bar{\mathcal{W}}_k$ share the same set of eigenvalues when $\mathcal{G}$ is structurally balanced.

Remark 4: The problem of leader group selection on structurally balanced signed graphs has been partially studied in [29], which requires the leaders to be selected from the same partitioned set (i.e., $\mathcal{V}_1$ or $\mathcal{V}_2$)

to ensure network controllability. Corollary 1 relaxes this constraint by allowing the leaders to be selected from different partitioned sets, as long as the corresponding unsigned graph is controllable under the selected leader group. Additionally, the discovered equivalence of controllability between $\mathcal{G}$ and $\bar{\mathcal{G}}$ enables the use of existing leader group selection methods developed for unsigned graphs [3], [4], [15].

Corollary 2: If $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is a signed acyclic graph, then there always exists an unsigned correspondence $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \bar{\mathcal{A}})$, such that the controllability Gramian $\mathcal{W}_k$ of $\mathcal{G}$ has the same matrix spectrum as that of $\bar{\mathcal{W}}_k$ of $\bar{\mathcal{G}}$.

Since acyclic graphs, e.g., tree or path graphs, are inherently structurally balanced [31], Corollary 2 is an immediate result of Theorem 4. Corollary 2 states that, instead of investigating the signed acyclic graph $\mathcal{G}$, the unsigned correspondence $\bar{\mathcal{G}}$ of $\mathcal{G}$ can be explored to enable leader group selection using the energy-related controllability metrics based on $\mathcal{W}_k$.

IV. Conclusion

In this article, the energy-related controllability of signed complex networks is characterized in terms of the graph Laplacian, thereby providing a means to select leaders with energy considerations. Although graph Laplacian has direct implications on the network topological properties, the potential connections between the energy-related controllability and its network topology have not been fully explored in this article. Future research will explore from topological perspectives (e.g., the graph resistances based on the inverse graph Laplacian) to facilitate the selection of optimal leader groups for energy-efficient network control.

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